SECTION 17.1: INTRODUCTION TO VECTOR FIELDS

RECALL: A vector-valued function takes real numbers to vectors. A function of several variables takes a point to a real number. Combining these ideas, we get the concept of a **vector field** which takes points to vectors.

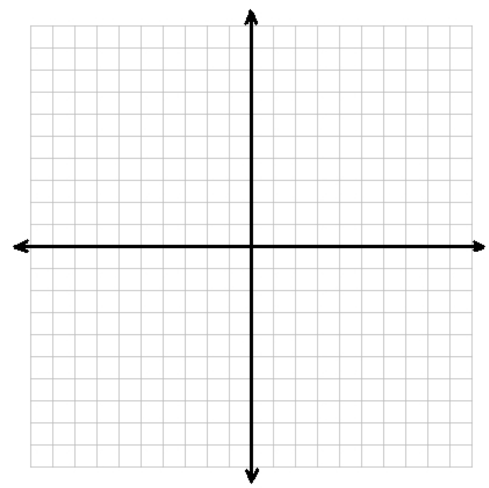
DEFINITION: A **vector field** is a function $\vec{F}$ which takes points to vectors. That is:

$$\vec{F}(x, y) = \langle M(x, y), N(x, y) \rangle = M(x, y)\,\hat{\imath} + N(x, y)\,\hat{\jmath}$$

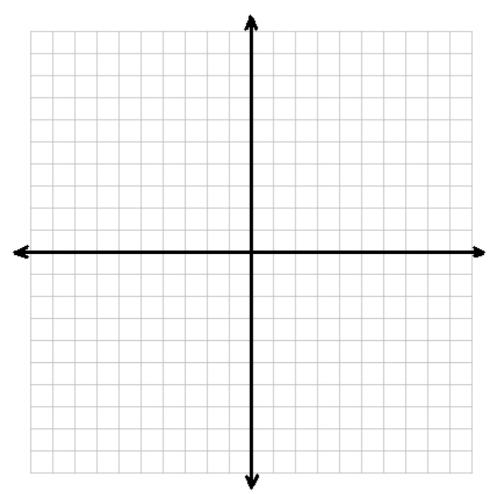
VISUALIZING VECTOR FIELDS: To visualize a field $\vec{F}$, we plot the vector $\vec{F}(x, y)$ with initial point (x, y).

EXAMPLE 1: With help from a graphing utility, sketch or otherwise describe the plot of the given vector field.

1. Sheer Field: $\vec{F}(x, y) = \langle y, 0 \rangle$



2. Channel Flow: $\vec{F}(x, y) = \langle 0, \sin(\pi x) \rangle$, $0 \leq x \leq \pi$



DEFINITION: Given a scalar function, $\phi(x, y)$, $\vec{F}(x, y) = \nabla\phi(x, y) = \langle \phi_x(x, y), \phi_y(x, y) \rangle$ is a **gradient field**.

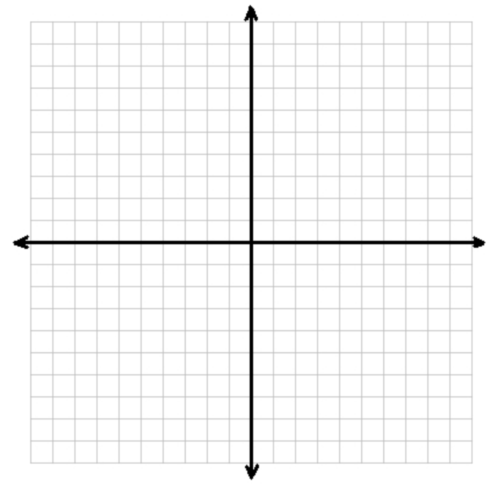
The function ϕ is called a **potential** for the field $\vec{F}$.

EXAMPLE 2: Find the gradient field $\vec{F} = \nabla\phi$ for $\phi(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$.

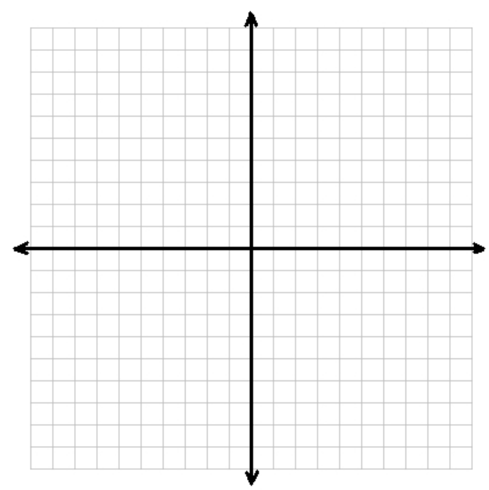
$$\vec{F}(x, y) = \nabla\phi(x, y) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

EXAMPLE 3: With help from a graphing utility, sketch or otherwise describe the plot of the given vector field.

1. Position Function (Radial Field): $\vec{F}(x, y) = \langle x, y \rangle$



2. Rotation Field: $\vec{F}(x, y) = \langle y, -x \rangle$



DEFINITIONS: A field $\vec{F}$ is called an:

- **Unit Vector Field** if $\left\| \vec{F}(x, y) \right\| = 1$ for all points in the domain of $\vec{F}$.

- **Inverse Square Field** if there is a $k > 0$ with $\left\| \vec{F}(x, y) \right\| = \dfrac{k}{x^2 + y^2}$ for all points in the domain of $\vec{F}$.

  NOTE: If $\vec{r}(x, y) = \langle x, y \rangle$, then $\|\vec{r}(x, y)\| = \sqrt{x^2 + y^2}$. Hence, $\left\| \vec{F}(x, y) \right\| = \dfrac{k}{\|\vec{r}(x, y)\|^2}$

  In words, this says the field strength, $\left\| \vec{F}(x, y) \right\|$ is inversely proportional to the square of the distance from the point (x, y) to the origin. Many important phenomena can be modeled using inverse square fields.

EXAMPLE 4: Let $\vec{F}(x,y) = \left\langle \dfrac{2x}{(x^2+y^2)^{3/2}}, \dfrac{2y}{(x^2+y^2)^{3/2}} \right\rangle$.

1. Show $\vec{F}$ is an inverse square field.

Ans: $\|\vec{F}(x,y)\| = \dfrac{2}{x^2+y^2}$

2. Find and simplify a formula for $\hat{U}(x,y) = \dfrac{\vec{F}(x,y)}{\left\|\vec{F}(x,y)\right\|}$ and show $\hat{U}$ is a unit vector field.

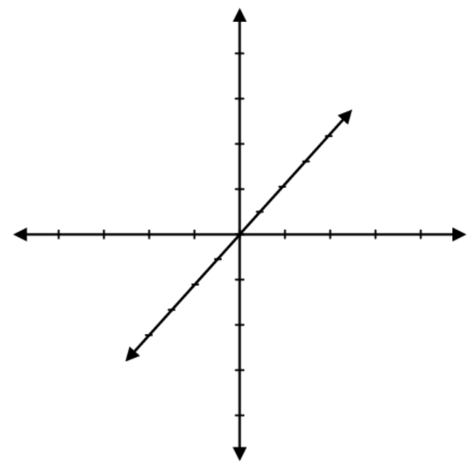
Ans: $\hat{U}(x,y) = \left\langle \dfrac{x}{(x^2+y^2)^{1/2}}, \dfrac{y}{(x^2+y^2)^{1/2}} \right\rangle$; $\|\hat{U}(x,y)\| = 1$.

EXTENSIONS TO THREE SPACE: A three dimensional vector field looks like:

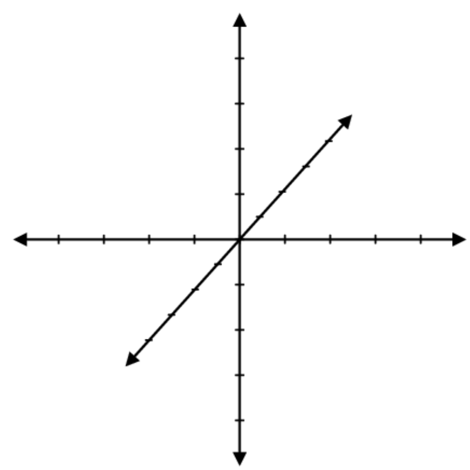
$$\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle = M(x,y,z)\,\hat{\imath} + N(x,y,z)\,\hat{\jmath} + P(x,y,z)\,\hat{k}$$

EXAMPLE 5: With help from a graphing utility, sketch or otherwise describe the plot of the given vector field.

1. Rotation Field: $\vec{F}(x,y,z) = \langle 2z - 3y, 3x - z, y - 2x\rangle$



2. Inverse Square Field: $\vec{F}(x,y,z) = \left\langle \dfrac{-x}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-y}{(x^2+y^2+z^2)^{3/2}}, \dfrac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$



EXAMPLE 6: Find the gradient field $\vec{F} = \nabla\phi$ for $\phi(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$.

$$\vec{F}(x,y,z) = \nabla\phi(x,y,z) = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

EXAMPLE 7: Pick your favorite nonzero vector $\vec{v}$. Let $\vec{r}(x,y,z) = \langle x, y, z\rangle$. Compute $\vec{F}(x,y,z) = \vec{v}\times\vec{r}$,

NOTE: This is a recipe for rotation fields!

HOMEWORK: Section 17.1: 9 - 53 every other odd.